1. (Ex 4.17)

Sol:

Here \( n = 82 \), \( \mu = 15 \text{ (sec)} \), \( \sigma = \sqrt{6} = 4 \text{ (sec)} \).

Convert the time into seconds, \( 20 \text{ min} = 20 \times 60 \text{ sec} = 1200 \text{ sec} \).

\[
P(\bar{S}_{82} < 1200) = P \left( \frac{\bar{S}_{82} - n\mu}{\sqrt{n} \sigma} < \frac{1200 - n\mu}{\sqrt{n} \sigma} \right)
\]

\[
= P (Z < \frac{1200 - 82 \times 15}{\sqrt{82} \times 4}) = P (Z < -0.83) = \Phi (-0.83)
\]

\[
= 0.2033 \quad \text{(By Table A4)}
\]

Continuity correction is not needed because the installation time of each jobs is already continuous variables.

2. (Ex 4.22)

Sol:

Denote the events \( A = \{ \text{Printer I} \} \) and \( B = \{ T < 1 \} \), where \( T \) is the printing time.

We know \( P(A) = 0.4 \), \( P(\bar{A}) = 0.6 \).

For Printer I with Exponential time, \( \mathbb{E}(T) = \frac{1}{\lambda} = 2 \), so \( \lambda = \frac{1}{2} \).

\[
P(B | A) = P(T < 1 | A) = 1 - e^{-\frac{1}{2} \times 1} = 1 - e^{-0.5} = 0.393.
\]

For Printer II with Uniform time,
the density of \( T \) is \( f(t) = \frac{1}{5} \) for \( t \in (0, 5) \), and
\[
p(B|\overline{A}) = P(T < 1 | \overline{A}) = \int_0^{1/5} dt = \frac{1}{5} = 0.2
\]

By the Bayes rule,
\[
P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B|A) \cdot P(A) + P(B|\overline{A}) \cdot P(\overline{A})}
\]
\[
= \frac{0.393 \times 0.4}{0.393 \times 0.4 + 0.2 \times 0.6}
\]
\[
= 0.567
\]

3. (Ex 6.2)

Sol:
(a) \( P^{(2)} = P \cdot P = \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.52 & 0.48 \\ 0.48 & 0.52 \end{pmatrix} \)

(b) There are 3 transitions between 5:30 and 8:30, thus we need to compute \( P^{(3)} \). The 3-step transition probability matrix is
\[
P^{(3)} = P^{(2)} \cdot P = \begin{pmatrix} 0.52 & 0.48 \\ 0.48 & 0.52 \end{pmatrix} \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.496 & \cdots \\ \cdots & \cdots \end{pmatrix}
\]
(there is no need to compute the entire matrix).

Hence, \( p_{ii}^{(3)} = 0.496 \).

\[ \times \]

4. (Ex 6.3)

**Set:**

(a)

Let "black" be state 1 and "brown" be state 2. Arranging the given probabilities in a matrix, get

\[ P = \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix} \]

\[ \times \]

(b)

There are 2 transitions between Rex and his grandchild.

\[ p_{21}^{(2)} = p_{21} \cdot p_{11} + p_{22} \cdot p_{21} = 0.2 \times 0.6 + 0.8 \times 0.2 = 0.12 + 0.16 \]

\[ = 0.28 \]

\[ \times \]
5. (Ex 6.4)

SOL:

(a) Let \( \chi(n) = 1 \) if \( n \)-th light is green; \( \chi(n) = 2 \) if it is red.

\[
P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}
\]

\[\star\]

(b) \( \pi_{12}^{(2)} = \pi_n \cdot p_{12} + \pi_2 \cdot p_{22} = 0.6 \times 0.4 + 0.4 \times 0.7 = 0.24 + 0.28 = 0.52 \)

\[\star\]

(c) We need to find \( \pi_2 = \lim_{n \to \infty} \chi(n) = 2 \). To find the steady-state distribution, solve the system \( \pi = \pi P \) with \( \pi_1 + \pi_2 = 1 \).

\[
\begin{pmatrix} \pi_1, & \pi_2 \end{pmatrix} = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} \pi_1, & \pi_2 \end{pmatrix}
\]

\[
0.6 \pi_1 + 0.3 \pi_2 = \pi_1
\]

\[
0.4 \pi_1 + 0.7 \pi_2 = \pi_2
\]

\[
\therefore 0.4 \pi_1 = 0.3 \pi_2
\]

\[
\pi_1 = \frac{3}{4} \pi_2
\]

\[
\pi_1 + \pi_2 = 1
\]

\[
\frac{3}{4} \pi_2 + \pi_2 = 1
\]

\[
\pi_2 = \frac{4}{7} = 0.5714 \]

\[\star\]
6. (Ex 6.5)

Sol:

Let "sunny" be state 1 and "rainy" be state 2. Write the transition probability matrix,

\[ P = \begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix} \]

(each row total is 1.)

April 1 next year is so many transitions away that we should better use the steady-state distribution. To find it, solve the system:

\[
\begin{align*}
\pi P &= \pi \\
\sum_{i=1}^{2} \pi_i &= 1
\end{align*}
\]

\[
\begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix} \begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix}
\]

\[
0.8 \pi_1 + 0.4 \pi_2 = \pi_1
\]

\[
0.2 \pi_1 + 0.6 \pi_2 = \pi_2
\]

\[
0.2 \pi_1 = 0.4 \pi_2
\]

\[
\pi_1 = 2 \pi_2
\]

\[
\therefore \pi_1 + \pi_2 = 1
\]

\[
2\pi_2 + \pi_2 = 1
\]

\[
\pi_2 = \frac{1}{3} = 0.3333
\]

\[\times\]

The probability that April 1 next year is rainy is \( \pi_2 = \frac{1}{3} = 0.3333 \).
7. (Ex 6.6)

SOL:

The transition probability matrix is given as

\[ P = \begin{pmatrix}
0.8 & 0.2 \\
0.1 & 0.9
\end{pmatrix} \]

Also, we have \( x(0) = 2 \) (idle mode) with probability 1,

i.e., \( P_0 = (0, 1) \).

(a)

\[ P_2 = P_0 \cdot P^2 = P_0 \cdot P \cdot P = (0, 1) \cdot \begin{pmatrix}
0.8 & 0.2 \\
0.1 & 0.9
\end{pmatrix} \cdot \begin{pmatrix}
0.8 & 0.2 \\
0.1 & 0.9
\end{pmatrix} \]

\[ = \begin{pmatrix}
0.1 & 0.9 \\
0.1 & 0.9
\end{pmatrix} \cdot \begin{pmatrix}
0.8 & 0.2 \\
0.1 & 0.9
\end{pmatrix} \]

\[ = \begin{pmatrix}
0.1 \times 0.8 + 0.9 \times 0.1 & 0.1 \times 0.2 + 0.9 \times 0.9 \\
0.1 \times 0.8 + 0.9 \times 0.1 & 0.1 \times 0.2 + 0.9 \times 0.9
\end{pmatrix} \]

\[ = \begin{pmatrix}
0.17 & 0.83 \\
0.17 & 0.83
\end{pmatrix} \]

That is, \( x(2) \), the state after 2 transitions, is busy with probability 0.17 and idle with probability 0.83.

(b) Solve the system of equations:

\[ \begin{align*}
\pi P &= \pi \\
\sum_{i=1}^{2} \pi_i &= 1
\end{align*} \]
\[
\begin{pmatrix}
\pi_1 \\ \pi_2
\end{pmatrix}
\cdot
\begin{pmatrix}
0.8 & 0.2 \\
0.1 & 0.9
\end{pmatrix}
= 
\begin{pmatrix}
\pi_1 \\ \pi_2
\end{pmatrix}
\]

\[0.8 \pi_1 + 0.1 \pi_2 = \pi_1\]
\[0.2 \pi_1 + 0.9 \pi_2 = \pi_2\]

\[0.2 \pi_1 = 0.1 \pi_2\]
\[\pi_2 = 2 \pi_1\]

\[\therefore \pi_1 + \pi_2 = 1\]
\[\pi_1 + 2 \pi_1 = 1\]
\[\pi_1 = \frac{1}{3}\]
\[\pi_2 = \frac{2}{3}\]

\[\pi = \begin{pmatrix}
\frac{1}{3} \\ \frac{2}{3}
\end{pmatrix}\]

8. (Ex 6.8)

Sol: The problem describes the following transition probability matrix,

\[
P = A \begin{pmatrix}
0 & 0 & \frac{c}{2} \\
\frac{a}{2} & 0 & \frac{c}{2} \\
\frac{b}{2} & 0 & 0
\end{pmatrix} 
\]

Find the steady-state distribution by solving:

\[
\begin{cases}
\pi P = \pi \\
\sum_{i=1}^{3} \pi_i = 1
\end{cases}
\]

\[\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}\]
For:
\[
\begin{pmatrix}
\pi_1 & \pi_2 & \pi_3
\end{pmatrix}
\begin{pmatrix}
0 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} \\
1 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
\pi_1 & \pi_2 & \pi_3
\end{pmatrix}
\]

\[
\begin{cases}
\frac{1}{2} \pi_2 + \pi_3 = \pi_1 \\
\frac{1}{2} \pi_1 = \pi_2 \\
\frac{1}{2} \pi_1 + \frac{1}{2} \pi_2 = \pi_3
\end{cases}
\]

Therefore:
\[
\pi_1 + \pi_2 + \pi_3 = 1
\]
\[
\pi_1 + \frac{1}{2} \pi_1 + \left(\frac{1}{2} \pi_1 + \frac{1}{2} \pi_2\right) = 1
\]
\[
2\pi_1 + \frac{1}{2} \pi_2 = 1
\]
\[
2\pi_1 + \frac{1}{2} \times \frac{1}{2} \pi_2 = 1
\]
\[
\frac{3}{4} \pi_1 = 1
\]
\[
\pi_1 = \frac{4}{3}
\]
\[
\pi_2 = \frac{1}{2} \pi_1 = \frac{2}{3}
\]
\[
\pi_3 = 1 - \pi_1 - \pi_2 = 1 - \frac{4}{3} - \frac{2}{3} = \frac{1}{3}
\]

\[
\pi = \left(\frac{4}{3}, \frac{2}{3}, \frac{1}{3}\right) \approx \left(0.4444, 0.2222, 0.3333\right)
\]

\(\times\)